Problem Set 2.


Problem 3. Arbitrage pricing.
Imagine that three states of the world are possible (that is, three possible outcomes in next month’s financial markets).

State 1: the Dow Jones Industrial Average goes up
State 2: the DJIA stays the same
State 3: the DJIA falls

Suppose also that there are three assets in the world—assets 1, 2, and 3. We will consider the payoff vectors of these assets, defined as follows:

\[ a_i = \begin{pmatrix} 
\text{value of asset } i \text{ if DJIA rises} \\
\text{value of asset } i \text{ if DJIA is unchanged} \\
\text{value of asset } i \text{ if DJIA falls}
\end{pmatrix} \]

Suppose that for the three assets,

\[ a_1 = \begin{pmatrix} 1 \\
0 \\
0
\end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\
1 \\
0
\end{pmatrix}, \quad \text{and } a_3 = \begin{pmatrix} 0 \\
0 \\
1
\end{pmatrix} \]

Let the current prices of the assets be given by the vector \( p = \begin{pmatrix} p_1 \\
p_2 \\
p_3 \end{pmatrix} \), that is, the current price of asset 1 is \( p_1 \), and so forth.

Define a portfolio \( x = \begin{pmatrix} x_1 \\
x_2 \\
x_3 \end{pmatrix} \) to be a vector of net purchases of the three assets.

That is, \( x_1 = 1 \) means that one unit of asset 1 is purchased, whereas \( x_1 = -2 \) means that two units of asset 1 are sold.

a. If you know something about financial markets, give a rough interpretation of what the three assets 1, 2, and 3 might represent.

b. Show that it is possible to purchase a portfolio of assets which has the total payoff \( \begin{pmatrix} 1 \\
1 \\
1 \end{pmatrix} \). (Show this by solving a matrix equation.) What is the total price of this portfolio?
c. Suppose there is a fourth asset on the market—namely, a riskless asset which pays \( a_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \). Suppose \( p_1 = 1, p_2 = 3, p_3 = 1, \) and \( p_4 = 3.5 \). Think of a get-rich-quick scheme. That is, describe how you can make limitless profits—regardless of what happens to the DJIA—by an appropriate choice of a portfolio \( x \). Such a scheme is called \textit{arbitrage}.

d. If \( p_1 = 1, p_2 = 3, \) and \( p_3 = 1, \) what price \( p_4 \) would make your arbitrage scheme impossible?

e. Suppose that prices in the market \textit{adjust} so that arbitrage becomes impossible. That is, if it is extremely profitable to sell asset \( i \), then many people attempt to sell it, driving its price down. Suppose \( p_1 = 1, p_2 = 3, \) and \( p_3 = 1, \) and suppose there is an asset 5 which pays off \( a_5 = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} \).

What must the price of asset 5 be, once the market adjusts to eliminate all opportunities for arbitrage?

f. Suppose that only the following four assets exist: assets 2, 3, and 4 which were already defined, and asset 6 which pays off \( a_6 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \). Suppose, as in (e), that prices adjust so that arbitrage becomes impossible. Is it possible to calculate \( p_6 \) if you only know \( p_2, p_3, \) and \( p_4? \) Explain why or why not (your explanation should include the words “vector space” or “invertible” or both...)